Propagation of Gaussian beams in a nonlinear medium

1998 J. Phys. A: Math. Gen. 31 1761

(http://iopscience.iop.org/0305-4470/31/7/010)

J. Phys. A: Math. Gen. **31** (1998) 1761–1769. Printed in the UK PII: S0305-4470(98)84395-3

Propagation of Gaussian beams in a nonlinear medium

Dagoberto S Freitas†, Jairo R de Oliveira† and Marco A de Moura‡

† Departamento de C. Exatas, Universidade Est. de F. de Santana, 44100-000, F de Santana, BA, Brazil

‡ Departamento de Fisica, Universidade Federal de Pernambuco, 50670-910, Recife, PE, Brazil

Received 20 May 1997, in final form 8 September 1997

Abstract. We investigate the propagation of cw (continuous wave) circularly symmetric Gaussian beams in a nonlinear saturable medium using a modified variational approach. We find the equations that describe the characteristics of the beam, solving them analytically in various regimes. We also determine the conditions under which these solutions may be stable in two transverse dimensions. Finally, we solve these equations numerically in the case of loss, comparing them with the lossless analytical solutions in the limit of small losses.

As is well known, strong beams of electromagnetic radiation propagating in a transparent medium may be subject to self-focusing as the result of a field-dependent refractive index. When the nonlinear focusing effects are balanced by the defocusing diffraction, a self-trapped mode of propagation is possible [1]. Various approximation schemes based on Gaussian ansatz functions have been devised, notably the paraxial ray theory [2, 3], which is known to ascribe too much importance to the central parts of the beams. It has been suggested [4, 5] that spatial diffraction leads to spectral features which are quantitatively and qualitatively different from those of the conventional self-phase modulation results. In particular, it has been claimed that, under certain circumstances, the blue might lead the red in the supercontinuum, thus opening the possibility for pulse compression without external grating.

In the theoretical treatment of these problems, much attention has been given to the variational approach [6–8]. A variational approach was employed in [8] deriving information about the various parameters that characterize the beam, which are qualitatively as well as quantitatively, in good agreement with numerical results [7]. This result invalidates the possibility of pulse compression without external grating which is erroneous and is only an artifact of the paraxial approximation. The discussion above does not consider the presence of the loss in the medium. It is well known that in real materials, the medium will not be purely transparent and the nonlinearity will not be of pure Kerr-law form, but will saturate.

Solitary wave solutions have been known to exist in a variety of nonlinear, dispersive media for many years. In the context of optical communications, Hasegawa and Tappert [9] first made the important observation that a pulse propagating in an optical fibre with Kerr-law nonlinearity can form an envelope soliton. This offered the potential for undistorted pulse transmission over very long distances. Just as a balance between self-phase modulation and group-velocity dispersion can lead to the formation of temporal solitons in single-mode fibres, it is also possible to have the analogous spatial solution, where diffraction and self-focusing can compensate for each other [10].

0305-4470/98/071761+09$19.50

A variational approach was employed by Anderson [11] in order to describe the main characteristics of the temporal soliton as determined by the cubic nonlinear Schrödinger (NLS) equation. These results were recently applied to the problem of propagation of cw Gaussian beams in a saturable medium with loss [12]. In this work [12] the diffraction is limited to one transverse solution. Hence we treat the problem in a more complete form, where we take into account the full cylindrical symmetry, i.e. two transverse directions.

The problem of describing the physical properties of dissipative systems has been the subject of lengthy discussions [13]. In order to develop a consistent classical formalism which includes dissipation, Herrera *et al* [14], proposed a variational principle which represents a modification of the Hamilton principle.

In this paper, we will analyse the dynamic interplay between nonlinearity and spatial diffraction through an optical saturated medium with loss using a variational approach in the form employed by Herrera *et al*. In the lossless case, exact analytical expressions for the behaviour of the beam are determined, and stable solutions for the spatial solitons in a Kerr-law medium with two transverse dimensions are obtained.

The starting point of our analysis is the conventional equation for the envelope of the circulary symmetry scalar field E through the medium with loss,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right) - 2\mathrm{i}k\frac{\partial E}{\partial z} - \mathrm{i}\alpha k E + \frac{2n_2K^2}{n_0}|E|^2E - \frac{2n_4K^2}{n_0}|E|^4E = 0 \qquad (1)$$

where r is the radial coordinate, z is the longitudinal coordinate, k is the linear wavenumber, and the refractive index n is assumed to be the form

$$n = n_0 - \frac{\mathrm{i}\alpha c}{2\omega} + n_2|E|^2 - n_4|E|^4$$

with n_0 the linear refractive index of the medium, α the medium loss, n_2 the third-order nonlinear coefficient and n_4 the fifth-order nonlinear coefficient.

Now we can handle equation (1) adequately in the form

$$\frac{\partial}{\partial r}\left(\mathrm{e}^{\alpha z}\frac{\partial L}{\partial E_r^*}\right) + \frac{\partial}{\partial z}\left(\mathrm{e}^{\alpha z}\frac{\partial L}{\partial E_z^*}\right) - \frac{\partial L}{\partial E^*}(\mathrm{e}^{\alpha z}L) = 0 \qquad (2)$$

where

$$L = r\left[\left|\frac{\partial E}{\partial r}\right|^2 - \mathrm{i}kr\left(E\frac{\partial E^*}{\partial z} - E^*\frac{\partial E}{\partial z}\right) - \frac{n_2k^2}{n_0}|E|^4 + \frac{2n_4k^2}{3n_0}|E|^6\right].$$

E^* is the complex conjugate of E and subindexes r, z are the differentiation with respect to r and z. L is the Lagrangian of the system without loss. Equation (2) is the Euler–Lagrange equation in the modified form that describes the propagation of the beam in the medium with loss, and can be written in the form of the modified Hamilton principle [14],

$$\delta\int_0^\infty\int_0^\infty \mathrm{e}^{\alpha z}L\,\mathrm{d}r\,\mathrm{d}z = 0. \qquad (3)$$

Assuming a trial functional of the form

$$E(z,r) = A(z)\exp\left[-\frac{1}{2}\frac{r^2}{a^2(z)} + \mathrm{i}b(z)r^2\right] \qquad (4)$$

and using the variational formulation, equation (3), we can integrate the r dependence explicitly to obtain

$$\delta\int_0^\infty \langle \mathrm{e}^{\alpha z}L\rangle \mathrm{d}z = 0 \qquad (5)$$

where

$$\langle e^{\alpha z} L\rangle = \left[(1+4b^2a^4)|A|^2 - \mathrm{i}k\left(A\frac{\mathrm{d}A^*}{\mathrm{d}z} - A^*\frac{\mathrm{d}A}{\mathrm{d}z}\right)a^2 \right.$$
$$\left. -2k|A|^2a^4\frac{\mathrm{d}b}{\mathrm{d}z} - \frac{n_2k^2a^2|A|^4}{2n_0} + \frac{2n_4k^2a^2|A|^6}{9n_0}\right]e^{\alpha z}. \tag{6}$$

Then, from the standard calculus, deriving $\langle e^{\alpha z} L\rangle$ with respect to A, A^*, a and b we obtain the following system of coupled ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}(a^2|A|^2) = -\alpha a^2|A|^2 \tag{7a}$$

$$\mathrm{i}ka^2\left(A^*\frac{\mathrm{d}A}{\mathrm{d}z} - \frac{\mathrm{d}A^*}{\mathrm{d}z}\right) = |A|^2\left(1+4b^2a^4 - 2ka^4\frac{\mathrm{d}b}{\mathrm{d}z} - \frac{n_2k^2a^2|A|^2}{n_0} + \frac{2n_4k^2a^2|A|^4}{3n_0}\right) \tag{7b}$$

$$b = -\frac{k}{2a}\frac{\mathrm{d}a}{\mathrm{d}z} \tag{7c}$$

$$\frac{\mathrm{d}^2a}{\mathrm{d}z^2} + \frac{1}{k^2a^3}\left(\frac{n_2k^2a^2|A|^2}{2n_0} - 1\right) - \frac{a_0^2}{k^2a^5}\left(\frac{4n_4k^2a^2|A|^4}{9n_0}\right) = 0. \tag{7d}$$

It is obvious that once equation (7*d*) is solved for $a(z)$, the other pulse characteristics are easily obtained from equations (7*a*)–(7*c*). In particular, if the longitudinal phase ϕ of the amplitude A is introduced by writing $A = |A|\exp(\mathrm{i}\phi)$, equation (7*b*) can be written as

$$ka^2\frac{\mathrm{d}\phi}{\mathrm{d}z} = 1 - \frac{3}{2}\left(\frac{n_2k^2a^2|A|^2}{2n_0}\right) + \frac{10}{4}\left(\frac{4n_4k^2a^2|A|^4}{9n_0}\right). \tag{8}$$

This system of equations has no analytic solutions. However, we initially look at the lossless case, by setting $\alpha = 0$, and show that in this case analytic solutions can be found.

In the case of lossless Gaussian beam propagation in a saturable nonlinear medium we focus our attention on equation (7*d*) since, once $a(z)$ is determined, $b(z)$ and $\phi(z)$ can also be found, and thus a knowledge of the dynamical behaviour of the cw beam can ascertained.

We normalize the spot width to the initial width by letting $y(z) = a(z)/a_0$, so that equation (7*d*) becomes

$$k^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \frac{1}{a_0^4y^3}(p-1) - \frac{P_s}{a_0^4y^5} = 0 \tag{9}$$

where

$$P = \frac{n_2k^2a_0^2A_0^2}{2n_0} \qquad P_s = \frac{4n_4k^2a_0^2A_0^4}{9n_0}$$

and we have used the fact that equation (7*a*) implies that $|A|^2a^2 = A_0^2a_0^2$ is independent of z. For this case it is not difficult to show, upon integration, that the spot width satisfies the dynamical equation

$$\frac{k^2}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 + \pi(y) = 0 \tag{10}$$

where

$$\pi(y) = \frac{\mu}{y^2} - \frac{\nu}{y^2} + \frac{\lambda}{y^4} - (\mu - \nu + \lambda) \tag{11}$$

and

$$\mu = \frac{1}{2a_0^4} \qquad \nu = \frac{p}{2a_0^4} \qquad \text{and} \qquad \lambda = \frac{p_s}{4a_0^4}.$$

The behaviour of the spot width is controlled by the nature of the potential function $\pi(y)$, since equation (10) can be interpreted as a *point mass* [12] under the influence of the potential. It is clear that as $y \to 0^+$, $\pi(y) \to \infty$, and that as $y \to \infty$, $\pi(y) \to -(\mu-\nu+\lambda)$. We note also that $\pi(1) = 0$. We are able to find solutions for the spot width by looking at the integral

$$\pm\frac{\sqrt{2}z}{k} = \int_1^y \frac{\mathrm{d}y}{\sqrt{-\pi(y)}} \tag{12}$$

and the type of solution will depend on the (μ, ν, λ) parameter space.

In figure 1, the potential function $\pi(y)$ is plotted for different regions of the (μ, ν, λ) parameter space. The nature of the potential function allows us to subdivide the parameter space as follows: (a) $\mu + \lambda - \nu > 0$; (b) $\mu + \lambda - \nu < 0$, with three subregions (i) $\mu + \lambda < \nu < 2\lambda + \mu$, (ii) $\nu > 2\lambda + \mu$ and (iii) $\nu = 2\lambda + \mu$; and (c) $\mu + \lambda - \nu = 0$. The dynamics of the beam in each of these regions will be now investigated.

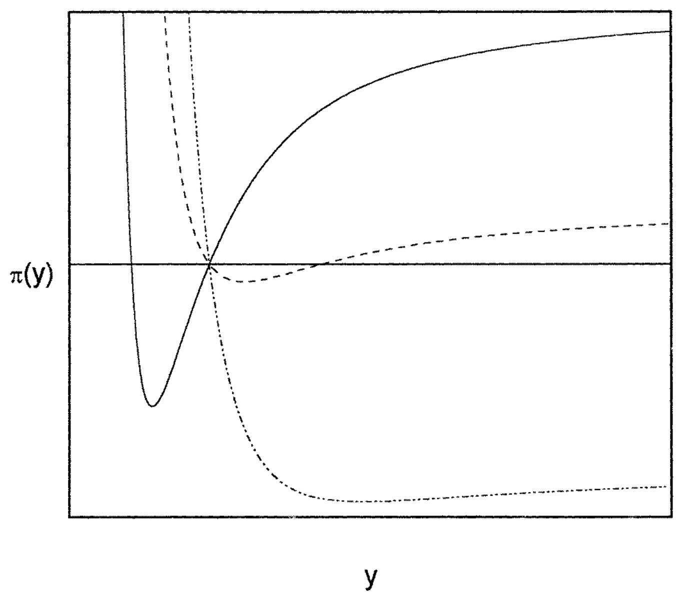

Figure 1. Qualitative plots of the potential function $\pi(y)$ in the (μ, ν, λ) parameter space: $\nu < \mu + \lambda$ (— · · —); $\mu + \lambda < \nu < \mu + 2\lambda$ (- - - -); $\nu > \mu + 2\lambda$ (——).

(a) $\mu + \lambda - \nu > 0$. For $\nu < \mu + \lambda$, the spot width once released at $y = 1$ will increase. The combined effect of linear diffraction and fifth-order nonlinearity overcome the cubic nonlinearity, so no stable solution exists.

In this case the solution of equation (9) is given by

$$\frac{z}{\sqrt{2}k} = \frac{(\mu + \lambda - \nu)^2}{\lambda^2}\left\{\left[1 - \frac{\lambda^2\Pi(\tilde{\mu}, 1, q)}{(\mu + \lambda - \nu)^2} + \frac{\lambda^4 F(\tilde{\mu}, q)}{(\mu + \lambda - \nu)^4}\right]\right\} \tag{13}$$

where $\Pi(\tilde{\mu}, 1, q)$ and $F(\tilde{\mu}, q)$ are the incomplete and the elliptic function of the first kind, respectively, with

$$\tilde{\mu} = \arcsin\sqrt{\frac{y^2 - 1}{y^2 - \lambda^2/(\mu + \lambda - \nu)^2}} \qquad q = \lambda/(\mu + \lambda - \nu).$$

As we can see, the Gaussian beam is diffracting as the spot width increases with increasing z as is shown in figure 1. For the special case in which $\mu = \nu$, the solution is given by

$$\frac{\sqrt{2\lambda}z}{k} = \frac{\sqrt{\pi}\,\Gamma(-1/4)}{4\Gamma(1/4)} + y[{}_2F_1(-1/4, 1/2, 3/4, 1/y^4)] \tag{14}$$

where ${}_2F_1$ is the hypergeometric function and Γ is the gamma function. In this case we notice that even with the balance between diffraction and cubic nonlinearity the beam keeps diffracting due to the presence of the fifth-order nonlinearity.

(b) $\mu + \lambda - \nu < 0$. In this region, we see that the potential function has a minimum (i.e. $\mathrm{d}\pi(y)/\mathrm{d}y = 0$) at $y_e = \sqrt{2\lambda/(\nu-\mu)}$ and zeros at $y_0 = 1$ and $y_1 = \sqrt{-\lambda/(\mu-\nu+\lambda)}$. We further note $\pi(y_e) \leqslant 0$. For this case the solution is given by

$$\frac{\sqrt{-2(\mu+\lambda-\nu)}}{2k} z = E(\tilde{\lambda}, p) \tag{15}$$

where $E(\tilde{\lambda}, p)$ is the elliptic function of the second kind, with

$$\tilde{\lambda} = \sqrt{\frac{1-y^2}{1-(\lambda^2/(\mu-\nu+\lambda)^2)}} \qquad \text{and} \qquad p = \sqrt{1 - \frac{\lambda^2}{(\mu-\nu+\lambda)^2}}.$$

(i) $\mu + \lambda < \nu < 2\lambda + \mu$. In this region, the potential function has two real roots with $y_1 > y_0$. Here, a beam with $y_0 = 1$ would initially diffract until it attains the largest possible value at y_1, at which stage self-focusing effects become dominant and the spot width decreases, returning to its minimum value y_0. As the spot width executes a homoclinic orbit, the resultant behaviour is oscillatory.

(ii) $\nu > 2\lambda + \mu$. Here $y_1 < y_0$, and thus the spot width initially decreases until it attains the minimum value y_1, at which point it becomes sufficiently small so that diffractive forces dominate and the spot width increases again until it reaches the maximum value y_0. Once again, the behaviour in this region is oscillatory.

(iii) $\nu = 2\lambda + \mu$. For the special case in which $y_e = 1$ and $\mathrm{d}\pi(1)/\mathrm{d}y = \pi(y_e)$, the potential well has degenerated into a single point and a particle released at this point will remain there. This translates into a beam propagating undistorted. There is an exact balance between the competitive forces of diffraction, self-focusing and saturation. The steady-state spot width is given by

$$\frac{1}{a_0} = k\left[\frac{n_2}{2n_0} - \frac{4n_4A_0^2}{9n_0}\right]^{1/2} A_0. \tag{16}$$

The corresponding phase shift is given by

$$2\phi(z) = k\left[-\frac{n_2A_0^2}{2n_0} - \frac{2n_4k^2A_0^4}{9n_0}\right] z. \tag{17}$$

(c) $\mu + \lambda - \nu = 0$. In the limiting case, the solution is given by

$$2\frac{\sqrt{2\lambda}}{k} z = y\sqrt{y^2-1} + \ln[y + \sqrt{y^2-1}] \tag{18}$$

and quite clearly the Gaussian beam is diffracting as the spot width y increases with increasing z.

Figure 2 illustrates the beam spot width variations as functions of the propagation distance for the different regions of the (μ, ν, λ) parameter space. For $\mu + \lambda - \nu > 0$ the spot width will increase diffraction with increasing z. For $\nu < 2\lambda + \mu$ the spot width will

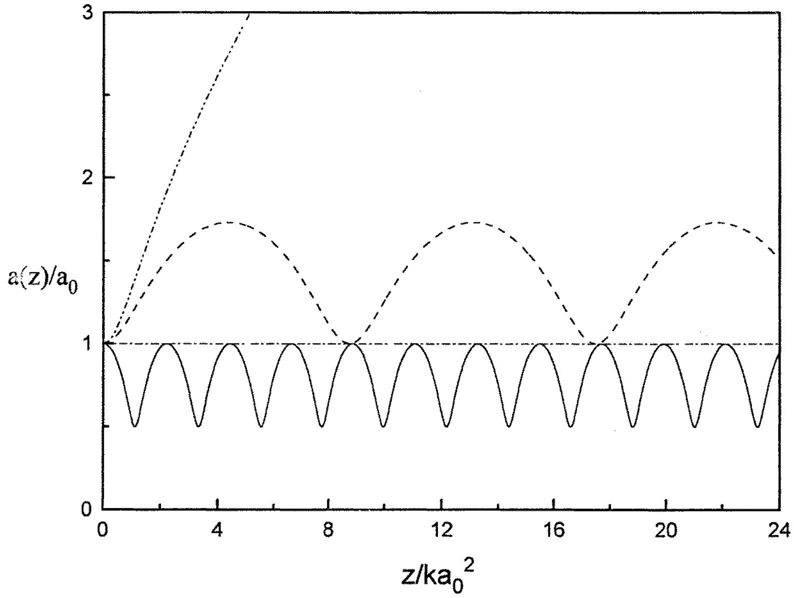

Figure 2. Propagation dependences of the normalized spot width in the (μ, ν, λ) parameter space: $\nu < \mu + \lambda$ (— · · —); $\mu + \lambda < \nu < \mu + 2\lambda$ (- - - -); $\nu > \mu + 2\lambda$ (——) and $\nu = \mu + 2\lambda$ (— · —). The initial width is a_0.

have oscillating diffraction. For $\nu > 2\lambda + \mu$ the spot width will have oscillating self-focusing and finally for $\nu = 2\lambda + \mu$ the beam will propagate undistorted. For $\mu + \lambda - \nu = 0$ the spot width will increase diffraction. For the phase shift $\phi(z)$ we can see from equation (8) that the phase is dependent on z through $a(z)$. As we can see, the behaviour of the phase will be associated with the behaviour of the width.

We are now in a position to investigate the stability of cw circularly symmetric Gaussian beams in this (μ, ν, λ) parameter space. According to equation (10), a perturbation from the equilibrium makes $\mathrm{d}^2 y/\mathrm{d}z^2 \neq 0$. If a variation in the beam parameters is such that it tends to re-establish the delicate balance between diffraction and self-focusing, the beam is said to be stable, otherwise it is unstable.

Stability may be determined by performing a Taylor expansion of the potential about the equilibrium point $y \simeq y_c$ [12], and linearizing the dynamical equation to find

$$k^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2}(y - y_e) + \frac{\mathrm{d}^2(y_e)}{\mathrm{d}y^2}(y - y_e) = 0 \qquad (19)$$

with

$$\frac{\mathrm{d}^2\pi(y_e)}{\mathrm{d}y^2} = \frac{(\nu - \mu)^3}{\lambda^2}.$$

For $\mu + \lambda - \nu < 0$ $(\nu > \mu)$ the quantity $\mathrm{d}^2\pi(y_e)/\mathrm{d}y^2$ is always positive, indicating stable equilibrium. The spot width will oscillate with the period given by

$$z_p = \frac{\lambda k \pi}{\sqrt{(\nu - \mu)^3}}.$$

These results show that the optical beam in a saturated nonlinear medium with two transverse dimensions is stable.

In the case of Gaussian propagation with loss in a saturable nonlinear medium we set $\alpha \neq 0$. As mentioned earlier, if $\alpha \neq 0$ then no exact solutions exist for the set of coupled equations given by equations (7*a*)–(7*d*). The equation characterizing the dynamics of the spot width in a saturable nonlinear medium with loss is given by

$$k^2 \frac{d^2 y}{dz^2} = \frac{2\mu}{y^3} - \frac{2\nu}{y^3} e^{-\alpha z} + \frac{4\lambda}{y^5} e^{-2\alpha z}. \tag{20}$$

Numerical solutions have recently been published for the Gaussian beam limited to one transverse dimension [12]. It was found that the fifth-order nonlinearity considerably modified the beam propagation and that the spot width of a cw Gaussian beam initially oscillates, passing through a series of maxima and minima before finally diffracting. The presence of the attenuation reduces the number of oscillations. We now investigate the dynamics of the beam propagation in a lossy medium by numerically solving equation (20) in the limit of small losses. The results of the numerical analysis are depicted in figures 3 and 4 where we have made a comparison with the lossless analytical solutions that we have obtained. We have considered the situations where the spot width undergoes initial decompression and also initial compression.

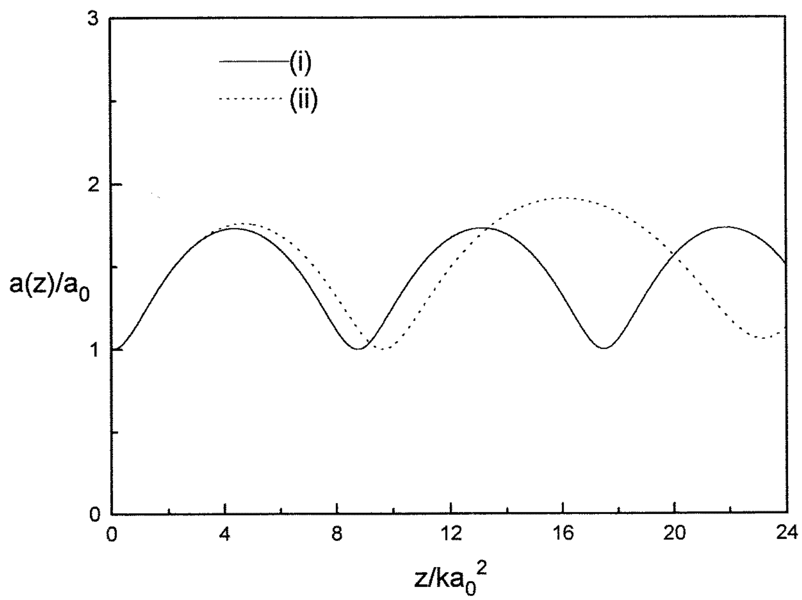

Figure 3. A comparison of the analytical solution for the variation of normalized spot width in the region $\mu + \lambda < \nu < \mu + 2\lambda$ with the numerical solution in the case of loss. (i) $\alpha k a_0^2 = 0$, (ii) $\alpha k a_0^2 = 0.015$. The beam width is a_0.

The numerical results show that the spot width of a cw Gaussian beam initially oscillates, passing through a series of maxima and minima before finally diffracting. The presence of the attenuation reduces the number of oscillations. These numerical results show that the two transversal Gaussian beams have identical behaviour to that of the one transversal Gaussian beam [12].

In conclusion, the propagation of Gaussian beams with cylindrical symmetry in a nonlinear saturable medium with and without loss has been analysed using a variational modified approach. This modified approach describes in a more consistent way the

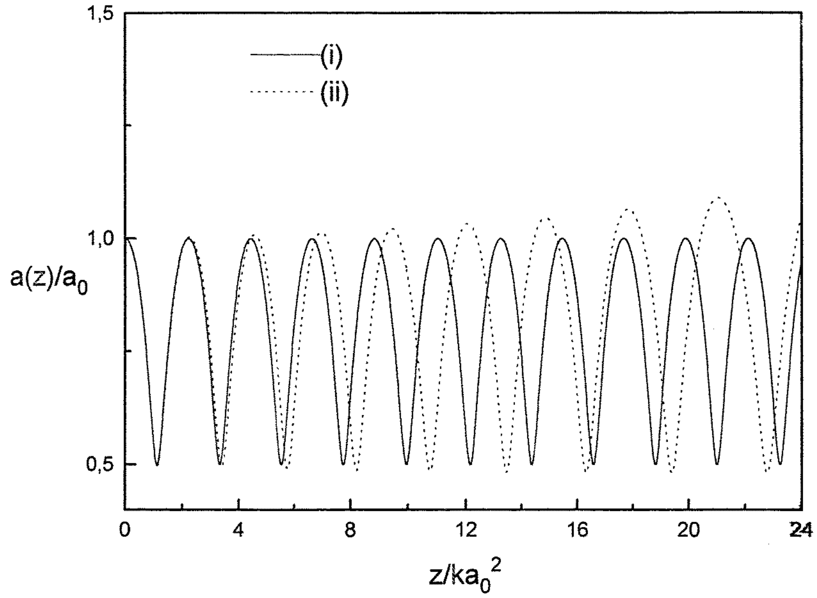

Figure 4. The same as in figure 3 but in the region $\nu < \mu + 2\lambda$.

behaviour of the beam in dissipative systems. In the lossless case, we were able to find exact analytic solutions for the behaviour of the spot width, and to determine conditions under which steady-state propagation was possible. Numerical solutions for Gaussian beams under dissipation were also considered, and these results show that the presence of the dissipation affects the nonlinearity and contributes to the diffraction of the beam.

Acknowledgments

Financial support for this research by CNPq, FINEP, CAPES and FACEPE, Brazilian agencies, is gratefully acknowledged. From October 1997, Marco A de Moura will be a visiting professor at Departamento de Biofisica e Radiobiologia, Universidade Federal de Pernambuco, Recife, 50.000, PE, Brazil.

References

[1] Chiao R Y, Garmire E and Yownes C H 1964 *Phys. Rev. Lett.* **13** 479
[2] Max C E 1976 *Phys. Fluids* **19** 74
[3] Sodha M S and Tripathi V K 1977 *Phys. Rev.* A **16** 201
[4] Manassah J T, Baldeck P L and Alfano R R 1988 *Opt. Lett.* **13** 1090
[5] Manassah J T, Baldeck P L and Alfano R R 1988 *Opt. Lett.* **13** 589
[6] Firth W J 1977 *Opt. Commun.* **22** 226
[7] Anderson D, Bonnedal M and Lisak M 1979 *Phys. Fluids* **22** 1838
Manassah J T and Gross B 1992 *Opt. Lett.* **17** 976
Sammut R A and Pask C 1991 *J. Opt. Soc. Am.* B **8** 395
[8] Kalsson M, Anderson D, Desaix M and Lisak M 1991 *Opt. Lett.* **16** 1973
[9] Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 142
[10] Zakharov V E and Sabat A B 1971 *Zh. Eksp. Teor. Fiz.* **61** 118 (Engl. transl. 1972 *Sov. Phys.–JETP* **34** 62)

[11] Anderson D 1983 *Phys. Rev.* A **27** 3135
[12] Jovanoski Z and Sammut R A 1994 *Phys. Rev.* E **50** 4087
[13] Ray J R 1979 *Am. J. Phys.* **47** 626
[14] Herrera L, Núñez L, Patiño A and Rago H 1986 *Am. J. Phys.* **54** 273